A subclass of paranormal including class of log-hyponormal and several related classes (Operator Inequalities and related topics)

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Abstract

This report is based on the following preprint:

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We shall introduce a new class "class A" given by an operator inequality which includes the class of log-hyponormal operators and is included in the class of paranormal operators. It turns out that our results contain another proof of Ando's result [3] which *every log-hyponormal operator is paranormal.* Moreover we shall introduce new classes related to class A operators and paranormal operators.

1 Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for a positive number p and *log-hyponormal* if T is invertible and $\log T^*T \geq \log TT^*$. p-Hyponormal and log-hyponormal operators were defined as extensions of hyponormal one, i.e., $T^*T \geq TT^*$, and also they have been studied by many authors, for instance, [1][2][5][8][13][14][17][18] and [19]. By the celebrated Löwner-Heinz inequality *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$"*, every p-hyponormal operator is q-hyponormal for $p \geq q > 0$. And every invertible p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function.

An operator T is *paranormal* if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. It has been studied by many authors, so there are too many to cite their references, for instance, [3][9] and [15]. Ando [3] proved the following result.

Theorem A.1 ([3]). *Every log-hyponormal operator is paranormal.*

In this paper, firstly we shall introduce a new class "class A" given by an operator inequality which properly includes the class of log-hyponormal operators and is properly included in the class of paranormal operators. It turns out that our results contain another proof of Theorem A.1. Secondly we shall introduce new classes related to class A operators and paranormal operators. Finally we shall give several examples to show that inclusion relations among these classes are all proper.

2 A subclass of paranormal operators including class of log-hyponormal

We shall introduce a new class of operators as follows:

Definition 1. *An operator T belongs to class A if*

$$|T^2| \geq |T|^2. \tag{2.1}$$

We would like to remark that class *"A"* is named after the *"absolute"* values of two operators $|T^2|$ and $|T|$ in (2.1). We call an operator T class A operator briefly if T belongs to class A. We obtain the following results on class A operators.

Theorem 1.
(1) *Every log-hyponormal operator is class A operator.*
(2) *Every class A operator is paranormal operator.*

Theorem 1 contains another proof of Theorem A.1. The following theorems and lemma play an important role in the proofs of the results in this paper.

Theorem B.1 ([6][10]). *Let A and B be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$.
(ii) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for all* $p \geq 0$.
(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p \geq 0$ *and* $r \geq 0$.

We remark that equivalence relation between (i) and (ii) is shown in [4].

Theorem B.2 (Hölder-McCarthy inequality [16]). *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*

(i) $(A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)}$ *for* $0 < r \leq 1$.
(ii) $(A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

As a slightly modification of [11, Lemma 1], we have the following lemma.

Lemma B.3. *Let A and B be invertible operators. Then*

$$(BAA^*B^*)^\lambda = BA(A^*B^*BA)^{\lambda-1}A^*B^*$$

holds for any real number λ.

Proof of Theorem 1.
Proof of (1). Suppose that T is log-hyponormal. T is log-hyponormal iff

$$\log |T|^2 \geq \log |T^*|^2. \tag{2.2}$$

By the equivalence between (i) and (ii) of Theorem B.1, (2.2) is equivalent to

$$|T|^{2p} \geq (|T|^p|T^*|^{2p}|T|^p)^{\frac{1}{2}} \quad \text{for all } p \geq 0. \tag{2.3}$$

Put $p = 1$ in (2.3), then we have

$$|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}}. \tag{2.4}$$

By Lemma B.3 and $|T^*|^2 = TT^*$, (2.4) holds iff

$$|T|^2 \geq |T|T(T^*|T|^2T)^{\frac{-1}{2}}T^*|T|$$

iff

$$(T^*|T|^2T)^{\frac{1}{2}} \geq T^*T, \tag{2.5}$$

so that

$$|T^2| \geq |T|^2,$$

that is, T is class A.

Proof of (2). Suppose that T is class A, i.e.,

$$|T^2| \geq |T|^2. \tag{2.1}$$

Then for every unit vector $x \in H$,

$$\begin{aligned}
\|T^2x\|^2 &= ((T^2)^*T^2x, x) \\
&= (|T^2|^2x, x) \\
&\geq (|T^2|x, x)^2 && \text{by (ii) of Theorem B.2} \\
&\geq (|T|^2x, x)^2 && \text{by (2.1)} \\
&= \|Tx\|^4.
\end{aligned}$$

Hence we have

$$\|T^2x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x \in H,$$

so that T is paranormal.

Whence the proof of Theorem 1 is complete. □

3 Several classes related to class A and paranormal

In this section, we shall discuss extensions of class A operators and paranormal operators. We shall introduce new classes of operators as follows:

Definition 2.
(1) *For each $k > 0$, an operator T is class $A(k)$ if*

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2. \tag{3.1}$$

(2) *For each $k > 0$, an operator T is absolute-k-paranormal if*

$$\left\||T|^k Tx\right\| \geq \left\|Tx\right\|^{k+1} \quad \textit{for every unit vector } x \in H. \tag{3.2}$$

An operator T is class A (resp. paranormal) if and only if T is class A(1) (resp. absolute-1-paranormal). Now we shall discuss the inclusion relations among these classes.

Theorem 2.
(1) *Every invertible class A operator is class $A(k)$ operator for $k \geq 1$.*
(2) *Every paranormal operator is absolute-k-paranormal operator for $k \geq 1$.*
(3) *For each $k > 0$, every class $A(k)$ operator is absolute-k-paranormal operator.*

We need the following theorem in order to give a proof of Theorem 2.

Theorem C.1 ([12]). *Let A and B be positive invertible operators such that $A^{\beta_0} \geq (A^{\frac{\beta_0}{2}} B^{\alpha_0} A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then the following inequality holds:*

$$A^{\beta} \geq (A^{\frac{\beta}{2}} B^{\alpha} A^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \quad \textit{for all } \alpha \geq \alpha_0 \textit{ and } \beta \geq \beta_0.$$

Proof of Theorem 2.
Proof of (1). Suppose that T is class A, i.e.,

$$|T^2| \geq |T|^2. \tag{2.1}$$

(2.1) holds if and only if

$$(T^*|T|^2T)^{\frac{1}{2}} \geq T^*T. \tag{2.5}$$

By Lemma B.3, (2.5) holds iff

$$T^*|T|(|T|TT^*|T|)^{\frac{-1}{2}}|T|T \geq T^*T$$

iff

$$|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}}. \tag{2.4}$$

Applying Theorem C.1 to (2.4), we have

$$|T|^{2k} \geq (|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}} \quad \text{for } k \geq 1. \tag{3.3}$$

By Lemma B.3 and $|T^*|^2 = TT^*$, (3.3) holds iff

$$|T|^{2k} \geq |T|^k T(T^*|T|^{2k}T)^{\frac{-1}{k+1}} T^*|T|^k \quad \text{for } k \geq 1$$

iff

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2 \quad \text{for } k \geq 1,$$

so that T is class A(k) for $k \geq 1$.

Proof of (2). Suppose that T is paranormal. Then for every unit vector $x \in H$ and $k \geq 1$,

$$\begin{aligned} \||T|^kTx\|^2 &= (|T|^{2k}Tx, Tx) \\ &\geq (|T|^2Tx, Tx)^k\|Tx\|^{2(1-k)} && \text{by (ii) of Theorem B.2} \\ &= \|T^2x\|^{2k}\|Tx\|^{2(1-k)} \\ &\geq \|Tx\|^{4k}\|Tx\|^{2(1-k)} && \text{by paranormality of } T \\ &= \|Tx\|^{2(k+1)}. \end{aligned}$$

Hence we have

$$\||T|^kTx\| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x \in H \text{ and } k \geq 1,$$

so that T is absolute-k-paranormal for $k \geq 1$.

Proof of (3). Suppose that T is class A(k) for $k > 0$, i.e.,

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2 \quad \text{for } k > 0. \tag{3.1}$$

Then for every unit vector $x \in H$,

$$\begin{aligned} \||T|^kTx\|^2 &= (T^*|T|^{2k}Tx, x) \\ &\geq ((T^*|T|^{2k}T)^{\frac{1}{k+1}}x, x)^{k+1} && \text{by (ii) of Theorem B.2} \\ &\geq (|T|^2x, x)^{k+1} && \text{by (3.1)} \\ &= \|Tx\|^{2(k+1)}. \end{aligned}$$

Hence we have

$$\||T|^kTx\| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x \in H,$$

so that T is absolute-k-paranormal for $k > 0$.

Whence the proof of Theorem 2 is complete. □

As further extensions of (2) and (3) of Theorem 2, we have the following two results.

Theorem 3. *Let T be an invertible class $A(k)$ operator for $k>0$. Then*

$$f(l) = (T^*|T|^{2l}T)^{\frac{1}{l+1}}$$

is increasing for $l \geq k > 0$, and the following inequality holds:

$$f(l) \geq |T|^2, \text{ i.e., } T \text{ is class } A(l) \text{ for } l \geq k > 0.$$

Theorem 4. *Let T be an absolute-k-paranormal operator for $k>0$. Then for every unit vector $x \in H$,*

$$F(l) = \left\| |T|^l Tx \right\|^{\frac{1}{l+1}}$$

is increasing for $l \geq k > 0$, and the following inequality holds:

$$F(l) \geq \left\| Tx \right\|, \quad \text{i.e., } T \text{ is absolute-l-paranormal for } l \geq k > 0.$$

Theorem 3 states the following: An operator function $f(l)$ asserts that every class A(k) operator is class A(l) for $l \geq k > 0$. Similarly, Theorem 4 states the following: A function of norm $F(l)$ asserts that every absolute-k-paranormal operator is absolute-l-paranormal for $l \geq k > 0$.

In order to give a proof of Theorem 3, we need the following theorem which is an extension of Theorem C.1.

Theorem C.2 ([12]). *Let A and B be positive invertible operators such that $A^{\beta_0} \geq (A^{\frac{\beta_0}{2}} B^{\alpha_0} A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then for fixed $\delta \geq -\beta_0$,*

$$f(\alpha, \beta) = A^{\frac{-\beta}{2}} (A^{\frac{\beta}{2}} B^{\alpha} A^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} A^{\frac{-\beta}{2}}$$

is a decreasing function of both α and β for $\alpha \geq \max\{\delta, \alpha_0\}$ and $\beta \geq \beta_0$.

Proof of Theorem 3. Suppose that T is class A(k) for $k > 0$, i.e.,

$$f(k) = (T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2. \tag{3.1}$$

By Lemma B.3, (3.1) holds iff

$$T^*|T|^k(|T|^k TT^*|T|^k)^{\frac{-k}{k+1}}|T|^k T \geq T^*T$$

iff

$$|T|^{2k} \geq (|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}}. \tag{3.4}$$

By applying Theorem C.2 to (3.4),

$$g(l) = |T|^{-l}(|T|^{l}|T^*|^2|T|^{l})^{\frac{l}{l+1}}|T|^{-l}$$

is decreasing for $l \geq k > 0$. And we have

$$\begin{aligned} g(l) &= |T|^{-l}(|T|^{l}|T^*|^2|T|^{l})^{\frac{l}{l+1}}|T|^{-l} \\ &= |T|^{-l}(|T|^{l}TT^*|T|^{l})^{\frac{l}{l+1}}|T|^{-l} \\ &= T(T^*|T|^{2l}T)^{\frac{-1}{l+1}}T^* && \text{by Lemma B.3} \\ &= T\{(T^*|T|^{2l}T)^{\frac{1}{l+1}}\}^{-1}T^* \\ &= T\{f(l)\}^{-1}T^*. \end{aligned}$$

Hence $f(l)$ is increasing for $l \geq k > 0$. Moreover,

$$(T^*|T|^{2l}T)^{\frac{1}{l+1}} = f(l) \geq f(k) \geq |T|^2,$$

that is, T is class A(l) for $l \geq k > 0$. Whence the proof of Theorem 3 is complete. □

Proof of Theorem 4. Suppose that T is absolute-k-paranormal for $k > 0$, i.e.,

$$\big\||T|^kTx\big\| \geq \big\|Tx\big\|^{k+1} \quad \text{for every unit vector } x \in H. \tag{3.2}$$

(3.2) holds if and only if

$$F(k) = \big\||T|^kTx\big\|^{\frac{1}{k+1}} \geq \big\|Tx\big\| \quad \text{for every unit vector } x \in H.$$

Then for every unit vector $x \in H$ and any l such that $l \geq k > 0$, we have

$$\begin{aligned} F(l) &= \big\||T|^lTx\big\|^{\frac{1}{l+1}} \\ &= (|T|^{2l}Tx, Tx)^{\frac{1}{2(l+1)}} \\ &\geq \left\{(|T|^{2k}Tx, Tx)^{\frac{l}{k}}\big\|Tx\big\|^{2(1-\frac{l}{k})}\right\}^{\frac{1}{2(l+1)}} && \text{by (ii) of Theorem B.2} \\ &= \left\{\big\||T|^kTx\big\|^{\frac{2l}{k}}\big\|Tx\big\|^{2(1-\frac{l}{k})}\right\}^{\frac{1}{2(l+1)}} \\ &\geq \left\{\big\|Tx\big\|^{\frac{2l(k+1)}{k}}\big\|Tx\big\|^{2(1-\frac{l}{k})}\right\}^{\frac{1}{2(l+1)}} && \text{by (3.2)} \\ &= \big\|Tx\big\|. \end{aligned}$$

Hence

$$F(l) = \big\||T|^lTx\big\|^{\frac{1}{l+1}} \geq \big\|Tx\big\| \quad \text{for every unit vector } x \in H \text{ and } l \geq k, \tag{3.5}$$

so that T is absolute-l-paranormal for $l \geq k > 0$.

Next we show that $F(l)$ is increasing for $l \geq k > 0$. For every unit vector $x \in H$ and any m and l such that $m \geq l \geq k > 0$,

$$\begin{aligned}
F(m) &= \left\| |T|^m Tx \right\|^{\frac{1}{m+1}} \\
&= (|T|^{2m}Tx, Tx)^{\frac{1}{2(m+1)}} \\
&\geq \left\{ (|T|^{2l}Tx, Tx)^{\frac{m}{l}} \|Tx\|^{2(1-\frac{m}{l})} \right\}^{\frac{1}{2(m+1)}} \quad \text{by (ii) of Theorem B.2} \\
&= \left\{ \left\| |T|^l Tx \right\|^{\frac{2m}{l}} \|Tx\|^{2(1-\frac{m}{l})} \right\}^{\frac{1}{2(m+1)}} \\
&\geq \left\{ \left\| |T|^l Tx \right\|^{\frac{2m}{l}} \left\| |T|^l Tx \right\|^{\frac{2}{l+1}(1-\frac{m}{l})} \right\}^{\frac{1}{2(m+1)}} \text{by (3.5)} \\
&= \left\| |T|^l Tx \right\|^{\frac{1}{l+1}} \\
&= F(l).
\end{aligned}$$

Hence $F(l)$ is increasing for $l \geq k > 0$. Whence the proof of Theorem 4 is complete. □

4 Examples

In this section, we shall give a characterization of absolute-k-paranormal operators. And by using this characterization, we shall give several examples showing that inclusion relations among the classes discussed in this paper are all proper.

Ando [3] proved the following Theorem D.1.

Theorem D.1 ([3]). *An operator T is paranormal if and only if*

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0 \quad \text{for all } \lambda > 0.$$

As a generalization of Theorem D.1, we have the following characterization of absolute-k-paranormal operators.

Theorem 5. *For each $k > 0$, an operator T is absolute-k-paranormal if and only if*

$$T^*|T|^{2k}T - (k+1)\lambda^k|T|^2 + k\lambda^{k+1} \geq 0 \quad \text{for all } \lambda > 0. \tag{4.1}$$

In fact Theorem 5 implies Theorem D.1 putting $k = 1$ in Theorem 5.

We cite the following well-known lemma in order to give a proof of Theorem 5.

Lemma D.2. *Let a and b be positive real numbers. Then*

$$a^\lambda b^\mu \leq \lambda a + \mu b \quad \text{for } \lambda > 0 \text{ and } \mu > 0 \text{ such that } \lambda + \mu = 1.$$

Proof of Theorem 5. Suppose that T is absolute-k-paranormal for $k>0$, i.e.,

$$\left\||T|^k Tx\right\| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x \in H. \tag{3.2}$$

(3.2) holds iff

$$\left\||T|^k Tx\right\|^{\frac{1}{k+1}} \|x\|^{\frac{k}{k+1}} \geq \|Tx\| \quad \text{for all } x \in H$$

iff

$$(T^*|T|^{2k}Tx, x)^{\frac{1}{k+1}}(x,x)^{\frac{k}{k+1}} \geq (|T|^2 x, x) \quad \text{for all } x \in H. \tag{4.2}$$

By Lemma D.2,

$$\begin{aligned}(T^*|T|^{2k}Tx, x)^{\frac{1}{k+1}}(x,x)^{\frac{k}{k+1}} &= \left\{\left(\frac{1}{\lambda}\right)^k (T^*|T|^{2k}Tx, x)\right\}^{\frac{1}{k+1}} \cdot \{\lambda(x,x)\}^{\frac{k}{k+1}} \\ &\leq \frac{1}{k+1}\frac{1}{\lambda^k}(T^*|T|^{2k}Tx, x) + \frac{k}{k+1}\lambda(x,x) \\ &\qquad \text{for all } x \in H \text{ and all } \lambda > 0,\end{aligned} \tag{4.3}$$

so (4.2) ensures the following (4.4) by (4.3).

$$\begin{aligned}\frac{1}{k+1}\frac{1}{\lambda^k}(T^*|T|^{2k}Tx, x) + \frac{k}{k+1}\lambda(x,x) &\geq (|T|^2 x, x) \\ \text{for all } x \in H \text{ and all } \lambda > 0.\end{aligned} \tag{4.4}$$

Conversely, (4.2) follows from (4.4) by putting $\lambda = \left\{\frac{(T^*|T|^{2k}Tx, x)}{(x,x)}\right\}^{\frac{1}{k+1}}$. (In case $(T^*|T|^{2k}Tx, x) = 0$, let $\lambda \to 0$.) Hence (4.2) is equivalent to (4.4), and (4.4) holds if and only if

$$T^*|T|^{2k}T - (k+1)\lambda^k|T|^2 + k\lambda^{k+1} \geq 0 \quad \text{for all } \lambda > 0, \tag{4.1}$$

so that the proof is complete. □

The following Proposition 6 is obtained by easy calculations, so we omit to describe these calculations.

Proposition 6. *Let* $K = \bigoplus_{n=-\infty}^{\infty} H_n$ *where* $H_n \cong H$*. For given positive operators* A, B *on* H*, define the operator* $T_{A,B}$ *on* K *as follows:*

$$T_{A,B} = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B & 0 & & & & \\ & & B & \boxed{0} & & & \\ & & & A & 0 & & \\ & & & & A & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{4.5}$$

where $\square$ *shows the place of the* $(0,0)$ *matrix element. Then the following assertions hold:*

(i) *$T_{A,B}$ is log-hyponormal if and only if A and B are invertible and*

$$\log A \geq \log B.$$

(ii) *For each $k > 0$, $T_{A,B}$ is class $A(k)$ if and only if*

$$(BA^{2k}B)^{\frac{1}{k+1}} \geq B^2.$$

(iii) *For each $k > 0$, $T_{A,B}$ is absolute-k-paranormal if and only if*

$$BA^{2k}B - (k+1)\lambda^k B^2 + k\lambda^{k+1} \geq 0 \quad \text{for all } \lambda > 0.$$

By using Proposition 6, we can give several examples to show that inclusion relations among these classes are all proper.

Example 1. *Let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong \mathbb{R}^2$. For given positive matrices A, B on $\mathbb{R}^2$, define the operator $T_{A,B}$ on K as* (4.5) *in Proposition 6. Then we have the following examples.*

We remark that the trace of a matrix X denotes $\operatorname{tr} X$ and the determinant of a matrix X denotes $\det X$.

(1) *An example of non-log-hyponormal, class A operator.*

Let

$$A = \begin{pmatrix} 17 & 7 \\ 7 & 5 \end{pmatrix}^2 \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^2.$$

Fujii, Furuta and Wang [7] shows that $\log A \not\geq \log B$ and $A^2 \geq (AB^2A)^{\frac{1}{2}}$ hold together. On the other hand, $A^2 \geq (AB^2A)^{\frac{1}{2}}$ holds iff $(BA^2B)^{\frac{1}{2}} \geq B^2$ holds by Lemma B.3. Therefore $T_{A,B}$ is non-log-hyponormal but class A by (i) and (ii) of Proposition 6.

(2) *An example of non-class A, class A(2), paranormal operator.*

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{23} \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}.$$

Then

$$(BA^2B)^{\frac{1}{2}} - B^2 = \begin{pmatrix} 0.17472\ldots & -3.1798\ldots \\ -3.1798\ldots & 11.770\ldots \end{pmatrix}.$$

Eigenvalues of $(BA^2B)^{\frac{1}{2}} - B^2$ are 12.585 ... and -0.64001 ... , so that $(BA^2B)^{\frac{1}{2}} \not\geq B^2$. So $T_{A,B}$ is non-class A by (ii) of Proposition 6.

On the other hand,

$$(BA^4B)^{\frac{1}{3}} - B^2 = \begin{pmatrix} 3.9481\ldots & -8.6943\ldots \\ -8.6943\ldots & 21.128\ldots \end{pmatrix}.$$

Eigenvalues of $(BA^4B)^{\frac{1}{3}} - B^2$ are 24.760 ... and 0.31608 ... , so that $(BA^4B)^{\frac{1}{3}} \geq B^2$. So $T_{A,B}$ is class A(2) by (ii) of Proposition 6.

Furthermore, for $\lambda > 0$, define $X_1(\lambda)$ as follows:

$$X_1(\lambda) = BA^2B - 2\lambda B^2 + \lambda^2 = \begin{pmatrix} 404 - 26\lambda + \lambda^2 & -576 + 24\lambda \\ -576 + 24\lambda & 844 - 26\lambda + \lambda^2 \end{pmatrix}.$$

Put $p_1(\lambda) = \operatorname{tr} X_1(\lambda)$ and $q_1(\lambda) = \det X_1(\lambda)$, then

$$\begin{aligned} p_1(\lambda) &= 2\lambda^2 - 52\lambda + 1248 \\ &= 2(\lambda - 13)^2 + 910 > 0 \end{aligned}$$

and

$$\begin{aligned} q_1(\lambda) &= (404 - 26\lambda + \lambda^2)(844 - 26\lambda + \lambda^2) - (-576 + 24\lambda)^2 \\ &= \lambda^4 - 52\lambda^3 + 1348\lambda^2 - 4800\lambda + 9200. \end{aligned}$$

By calculation,

$$\begin{aligned} q_1'(\lambda) &= 4\lambda^3 - 156\lambda^2 + 2696\lambda - 4800 \\ &= 4(\lambda - 2)(\lambda^2 - 37\lambda + 600) \\ &= 4(\lambda - 2)\left\{\left(\lambda - \frac{37}{2}\right)^2 + \frac{1031}{4}\right\}. \end{aligned}$$

So $q_1'(\lambda) = 0$ iff $\lambda = 2$, that is, $q_1(\lambda) \geq q_1(2) = 4592 > 0$ for all $\lambda > 0$. Hence $X_1(\lambda) \geq 0$ for all $\lambda > 0$ since $\operatorname{tr} X_1(\lambda) = p_1(\lambda) > 0$ and $\det X_1(\lambda) = q_1(\lambda) > 0$ for all $\lambda > 0$. Therefore $T_{A,B}$ is paranormal by (iii) of Proposition 6.

(3) *An example of non-class A(2), absolute-2-paranormal operator.*

Let

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 20 \end{pmatrix}^{\frac{1}{4}} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}.$$

Then

$$(BA^4B)^{\frac{1}{3}} - B^2 = \begin{pmatrix} -0.0091543\ldots & 0.44289\ldots \\ 0.44289\ldots & 1.2774\ldots \end{pmatrix}.$$

Eigenvalues of $(BA^4B)^{\frac{1}{3}} - B^2$ are 1.4151 ... and -0.14687 ... , so that $(BA^4B)^{\frac{1}{3}} \not\geq B^2$. So $T_{A,B}$ is non-class A(2) by (ii) of Proposition 6.

On the other hand, for $\lambda > 0$, define $X_2(\lambda)$ as follows:

$$X_2(\lambda) = BA^4B - 3\lambda^2B^2 + 2\lambda^3 = \begin{pmatrix} 24 - 8\sqrt{3} - 6\lambda^2 + 2\lambda^3 & -12 + 3\lambda^2 \\ -12 + 3\lambda^2 & 24 + 8\sqrt{3} - 6\lambda^2 + 2\lambda^3 \end{pmatrix}.$$

Put $p_2(\lambda) = \operatorname{tr} X_2(\lambda)$ and $q_2(\lambda) = \det X_2(\lambda)$, then

$$p_2(\lambda) = 4\lambda^3 - 12\lambda^2 + 48$$

and

$$\begin{aligned} q_2(\lambda) &= (24 - 8\sqrt{3} - 6\lambda^2 + 2\lambda^3)(24 + 8\sqrt{3} - 6\lambda^2 + 2\lambda^3) - 64 \\ &= 4\lambda^6 - 24\lambda^5 + 27\lambda^4 + 96\lambda^3 - 216\lambda^2 + 240. \end{aligned}$$

We easily obtain $p_2(\lambda) > 0$ for all $\lambda > 0$. And we have

$$\begin{aligned} q_2'(\lambda) &= 24\lambda^5 - 120\lambda^4 + 108\lambda^3 + 288\lambda^2 - 432\lambda \\ &= 12\lambda(\lambda - 2)(2\lambda^3 - 6\lambda^2 - 3\lambda + 18). \end{aligned}$$

So $q_2'(\lambda) = 0$ iff $\lambda = 0, 2$ since $2\lambda^3 - 6\lambda^2 - 3\lambda + 18 > 0$ for all $\lambda > 0$ by an easy calculation, that is, $q_2(\lambda) \geq q_2(2) = 64 > 0$ for all $\lambda > 0$. Hence $X_2(\lambda) \geq 0$ for all $\lambda > 0$ since $\operatorname{tr} X_2(\lambda) = p_2(\lambda) > 0$ and $\det X_2(\lambda) = q_2(\lambda) > 0$ for all $\lambda > 0$. Therefore $T_{A,B}$ is absolute-2-paranormal by (iii) of Proposition 6.

(4) *An example of non-paranormal, absolute-2-paranormal operator.*

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then for $\lambda > 0$, define $X_3(\lambda)$ as follows:

$$X_3(\lambda) = BA^2B - 2\lambda B^2 + \lambda^2 = \begin{pmatrix} 8 - 8\lambda + \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}. \tag{4.6}$$

Put $\lambda = 4$ in (4.6), then

$$X_3(4) = \begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix} \not\geq 0.$$

So $T_{A,B}$ is non-paranormal by (iii) of Proposition 6.

On the other hand, for $\lambda > 0$, define $X_4(\lambda)$ as follows:

$$X_4(\lambda) = BA^4B - 3\lambda^2B^2 + 2\lambda^3 = \begin{pmatrix} 80 - 12\lambda^2 + 2\lambda^3 & 0 \\ 0 & 2\lambda^3 \end{pmatrix}.$$

By an easy calculation, $80 - 12\lambda^2 + 2\lambda^3 > 0$ for all $\lambda > 0$. So $X_4(\lambda) \geq 0$ for all $\lambda > 0$, that is, $T_{A,B}$ is absolute-2-paranormal by (iii) of Proposition 6. □

5 Remarks

Remark 1. An operator T is *normaloid* if $\|T^n\| = \|T\|^n$ for all positive integers n. It is known that every paranormal operator is normaloid [9][15]. As a generalization of this result, we obtain the following Theoren 7.

Theorem 7. *If an operator T is absolute-k-paranormal for some $k > 0$, then T is normaloid.*

Proof of Theorem 7. In case T is absolute-k-paranormal for some $0 < k < 1$, T is paranormal by Theorem 4, so that T is normaloid as shown in [9] and [15]. So we consider only the case of $k \geq 1$. Suppose T is absolute-k-paranormal for some $k \geq 1$, i.e.,

$$\||T|^k Tx\| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x \in H. \tag{3.2}$$

(3.2) holds iff

$$\||T|^k Tx\| \|x\|^k \geq \|Tx\|^{k+1} \quad \text{for all } x \in H. \tag{5.1}$$

We shall show that

$$\|T^n\| = \|T\|^n \tag{5.2}$$

for all positive integers n. When $n = 1$, (5.2) is always holds. Assume that (5.2) holds for some positive integer n. Then for every unit vector $x \in H$, we have

$$\begin{aligned}
\|T^n x\|^{k+1} &= \|T \cdot T^{n-1}x\|^{k+1} \\
&\leq \||T|^k T \cdot T^{n-1}x\| \cdot \|T^{n-1}x\|^k \qquad \text{by (5.1)} \\
&\leq \||T|^{k-1}\| \cdot \||T|T^n x\| \cdot \|T^{n-1}\|^k \\
&\leq \|T\|^{k-1} \cdot \|T^{n+1}x\| \cdot \|T\|^{(n-1)k} \\
&\leq \|T^{n+1}\| \cdot \|T\|^{nk-1},
\end{aligned}$$

that is,

$$\|T^{n+1}\| \cdot \|T\|^{nk-1} \geq \|T^n\|^{k+1}. \tag{5.3}$$

By the assumption (5.2) for n, (5.3) holds if and only if

$$\|T^{n+1}\| \cdot \|T\|^{nk-1} \geq \|T\|^{n(k+1)},$$

that is $\|T^{n+1}\| \geq \|T\|^{n+1}$, so that $\|T^{n+1}\| = \|T\|^{n+1}$.

Therefore $\|T^n\| = \|T\|^n$ for all positive integers n by induction. Hence the proof of Theorem 7 is complete. □

We can give an example to show that inclusion relations between absolute-k-paranormal and normaloid are proper.

Example 2. *There exists a non-absolute-k-paranormal for any $k > 0$ and normaloid operator.*

Let

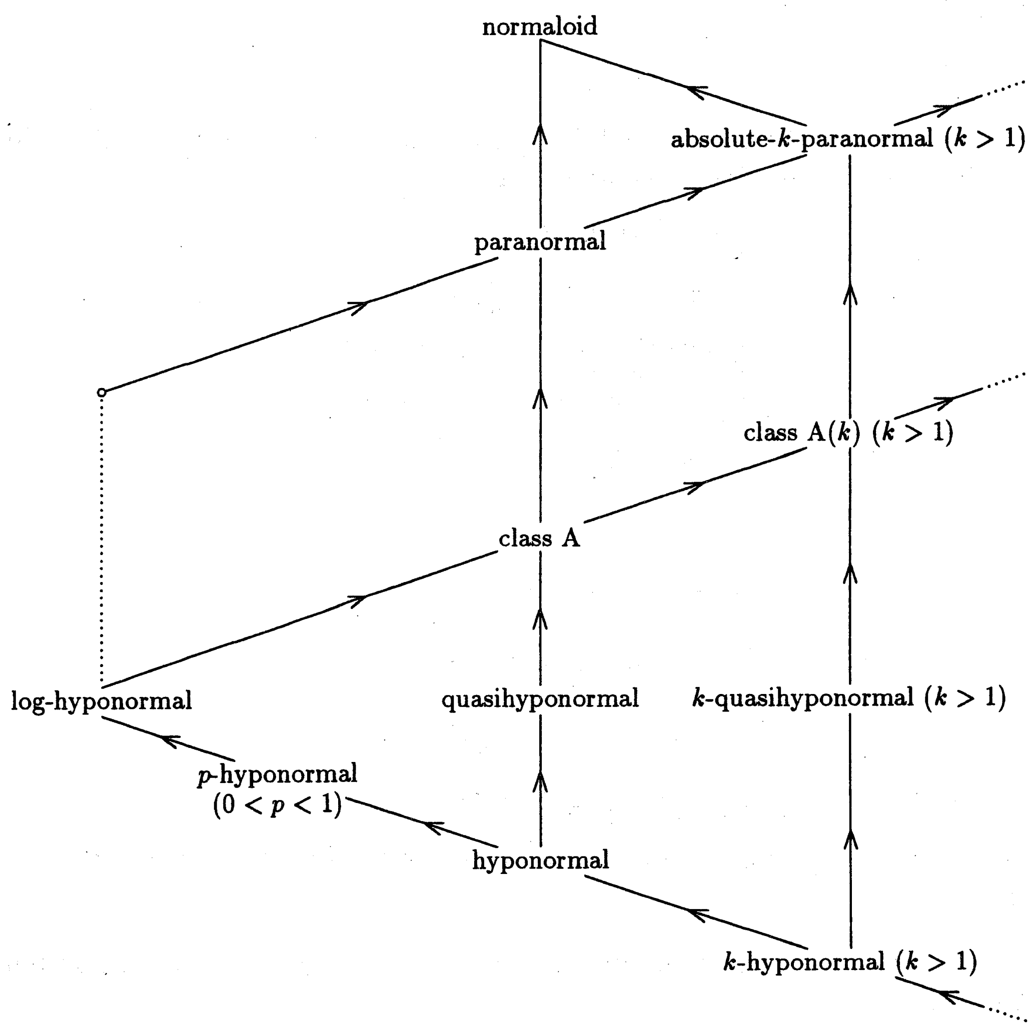
$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\|T^n\| = \|T\|^n$ for all positive integers n by an easy calculation. However, the relation $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ does not hold for the unit vector $e_2 = (0,1,0)$ since

$$|T|^k T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence T is non-absolute-k-paranormal for any $k > 0$. □

Remark 2. The following diagram expresses the inclusion relations among the classes discussed in this paper.



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